

Functions

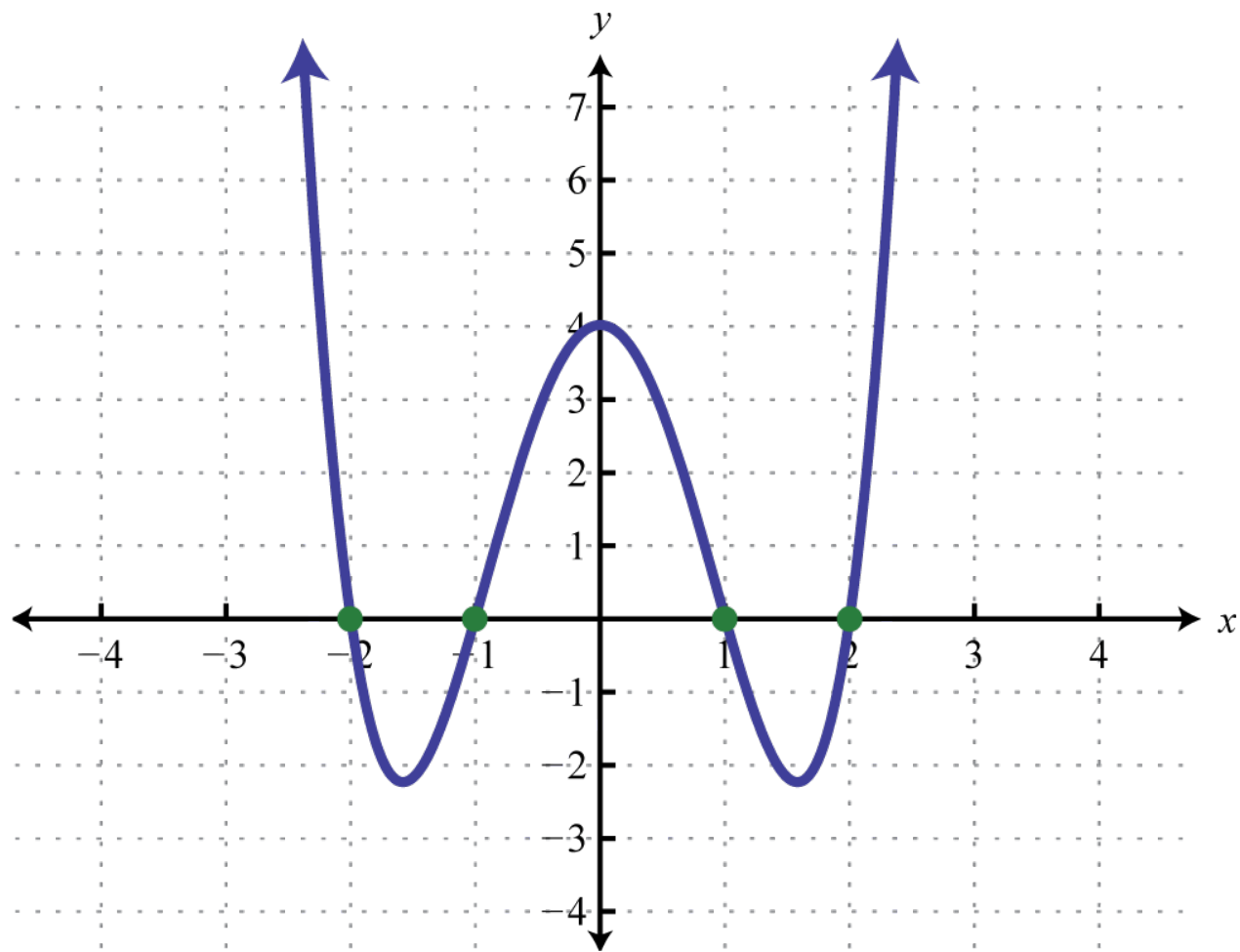
Part One

Outline for Today

- ***What is a Function?***
 - It's more nuanced than you might expect.
- ***Domains and Codomains***
 - Where functions start, and where functions end.
- ***Defining a Function***
 - Expressing transformations compactly.
- ***Special Classes of Functions***
 - Useful types of functions you'll encounter IRL.
- ***Proofs on First-Order Definitions***
 - A key skill.

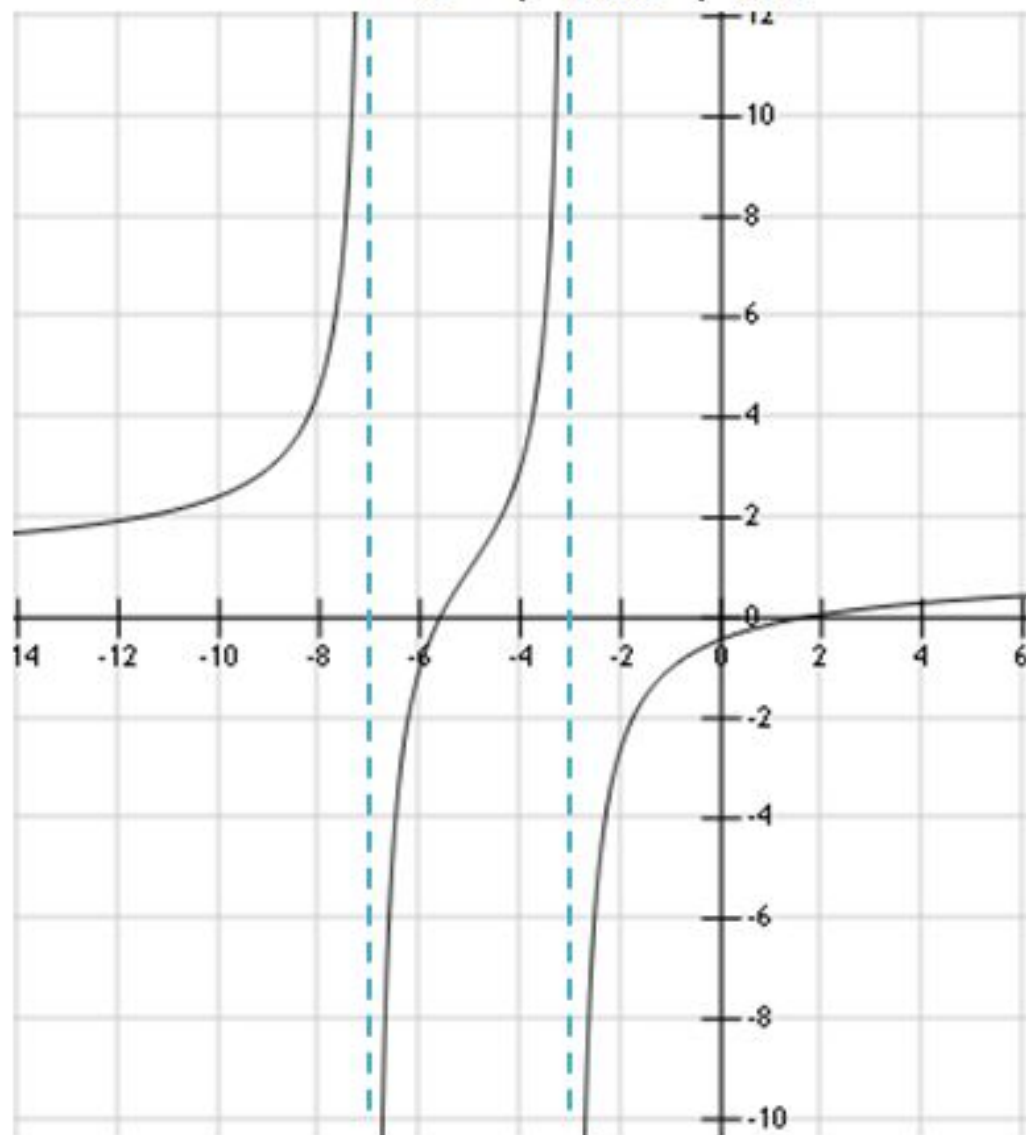
What is a function?

Functions, High-School Edition



$$f(x) = x^4 - 5x^2 + 4$$

$$f(x) = \frac{x^2 + 4x - 9}{x^2 + 10x + 21}$$



Functions, High-School Edition

- In high school, functions are usually given as objects of the form

$$f(x) = \frac{x^3 + 3x^2 + 15x + 7}{1 - x^{137}}$$

- What does a function do?
 - It takes in as input a real number.
 - It outputs a real number
 - ... except when there are vertical asymptotes or other discontinuities, in which case the function doesn't output anything.

Functions, CS Edition

```
int flipUntil(int n) {  
    int numHeads = 0;  
    int numTries = 0;  
  
    while (numHeads < n) {  
        if (randomBoolean()) {  
            numHeads++;  
        }  
        numTries++;  
    }  
  
    return numTries;  
}
```

Functions, CS Edition

- In programming, functions
 - might take in inputs,
 - might return values,
 - might have side effects,
 - might never return anything,
 - might crash, and
 - might return different values when called multiple times.

What's Common?

- Although high-school math functions and CS functions are pretty different, they have two key aspects in common:
 - They take in inputs.
 - They produce outputs.
- In math, we like to keep things easy, so that's pretty much how we're going to define a function.

High-Level Intuition:

A function is an object f that takes in exactly one input x and produces exactly one output $f(x)$.



(This is not definition. It's just to help you build and intuition.)

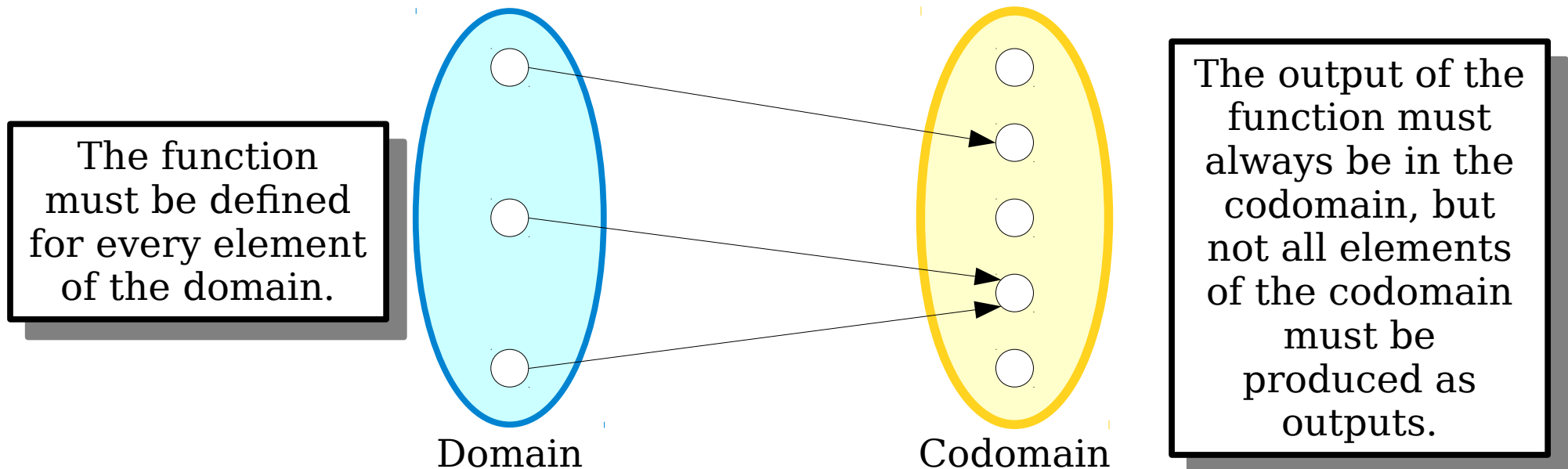
In mathematics, functions are ***deterministic***.
That is, given the same input, a function must
always produce the same output.

The following is a perfectly valid piece of
C++ code, but it's not a valid function under
our definition:

```
int randomNumber(int numOutcomes) {  
    return rand() % numOutcomes;  
}
```

Domains and Codomains

- Every function f has two sets associated with it: its **domain** and its **codomain**.
- A function f can only be applied to elements of its domain. For any x in the domain, $f(x)$ belongs to the codomain.



Domains and Codomains

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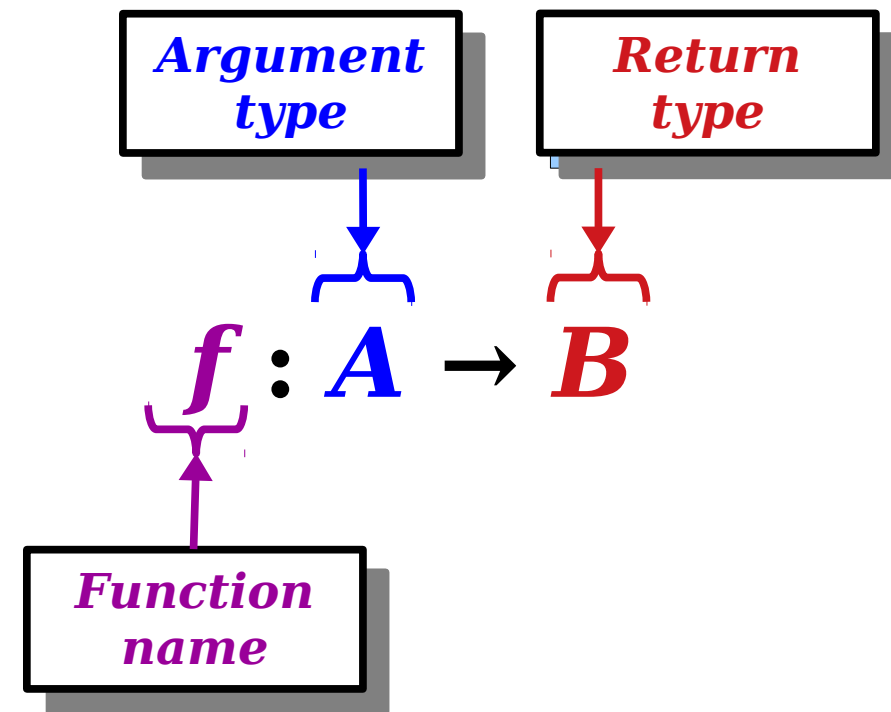
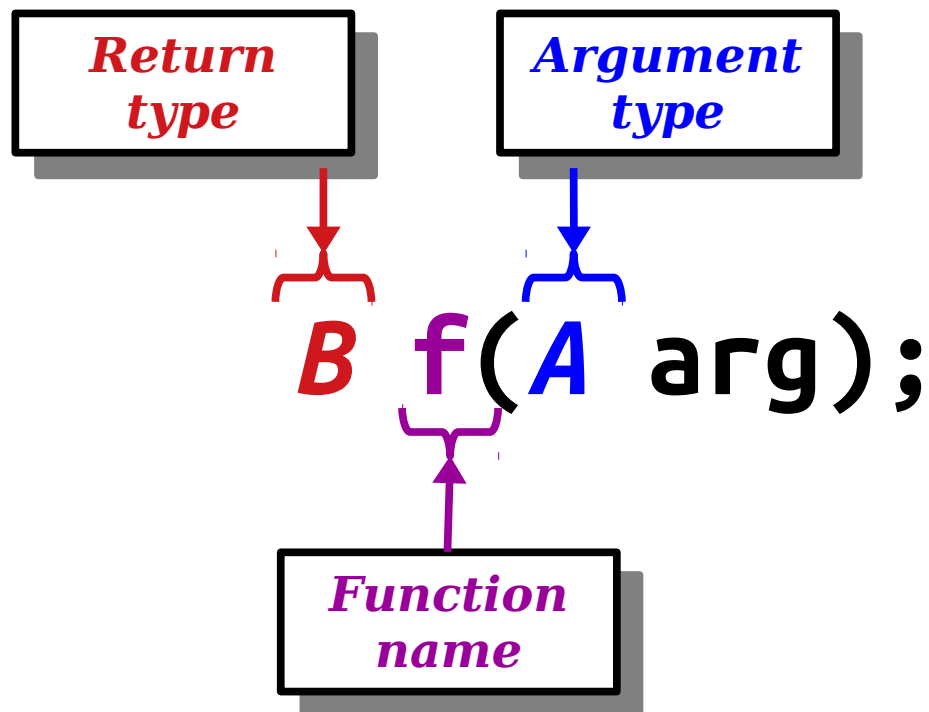
The **domain** of this function is \mathbb{R} . Any real number can be provided as input.

The **codomain** of this function is \mathbb{R} . Everything produced is a real number, but not all real numbers can be produced.

```
double absoluteValueOf(double x) {  
    if (x >= 0) {  
        return x;  
    } else {  
        return -x;  
    }  
}
```

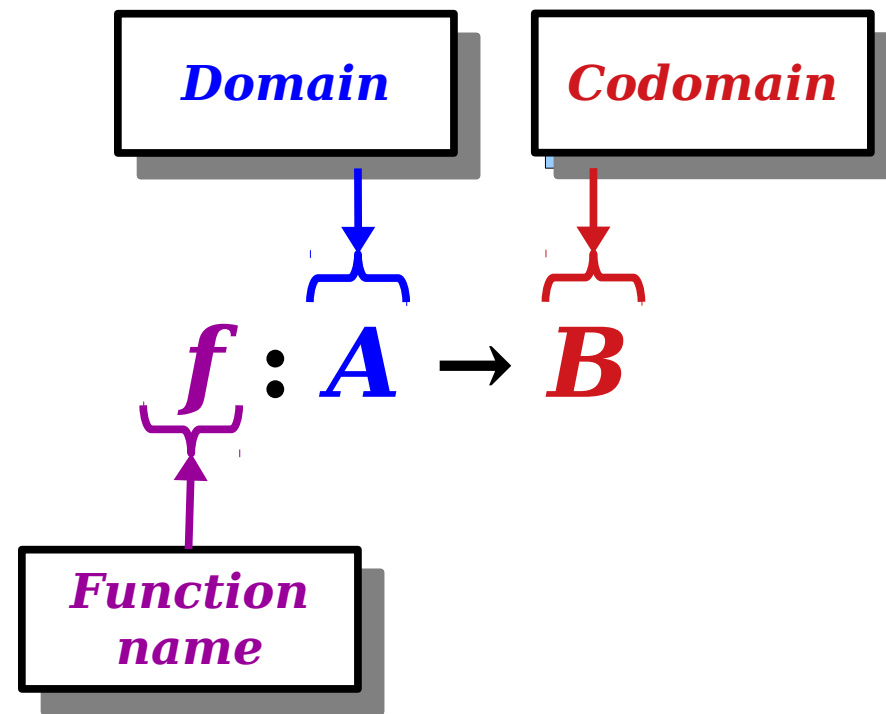
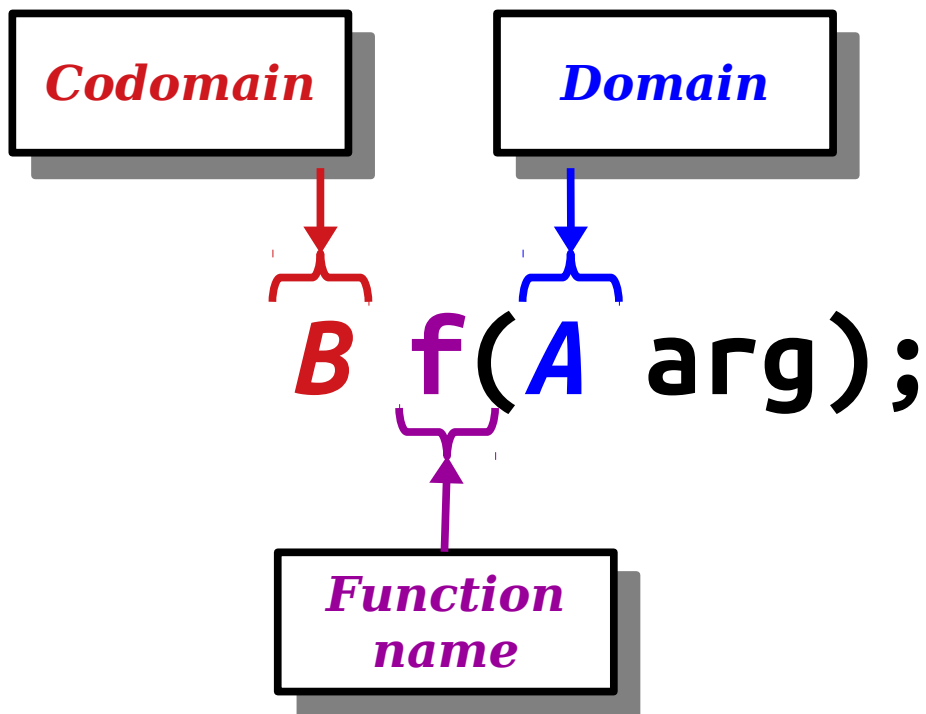
Domains and Codomains

- If f is a function whose domain is A and whose codomain is B , we write $f : A \rightarrow B$.
- Think of this like a “function prototype” in C++.



Domains and Codomains

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Some Observations

- Usually, when working with functions, you pick the domain and codomain before defining the rule for the function.
 - Think programming: you usually know what types of things you're working with before you know how they work.
- In mathematics, all functions take in exactly one argument: an element of the domain.
 - If you're clever, you can get two or more arguments to a function while still obeying this rule. Chat with me after class to learn more!
- In mathematics, functions are ***deterministic*** and can't behave randomly.
 - If you're clever, you can get functions that kinda sorta ish look random. Chat with me after class to learn more!

The Official Rules for Functions

- Formally speaking, we say that $f : A \rightarrow B$ if the following two rules hold.
- First, f must obey its domain/codomain rules:

$$\forall a \in A. \exists b \in B. f(a) = b$$

(“Every input in A maps to some output in B .”)

- Second, f must be deterministic:

$$\forall a_1 \in A. \forall a_2 \in A. (a_1 = a_2 \rightarrow f(a_1) = f(a_2))$$

(“Equal inputs produce equal outputs.”)

- If you’re ever curious about whether something is a function, look back at these rules and check! For example:
 - Can a function have an empty domain?
 - Can a function have an empty codomain?

Defining Functions

Defining Functions

- To define a function, you need to
 - specify the domain,
 - specify the codomain, and
 - give a **rule** used to evaluate the function.
- All three pieces are necessary.
 - We need to domain to know what the function can be applied to.
 - We need to codomain to know what the output space is.
 - We need the rule to be able to evaluate the function.
- There are many ways to do this. Let's go over a few examples.



*White-Tailed
Kite*

*Anna's
Hummingbird*

*Red-Shouldered
Hawk*

Functions can be defined as a ***picture***.
Draw the domain and codomain explicitly.
Then, add arrows to show the outputs.

$$f : \mathbb{Z} \rightarrow \mathbb{Z}, \text{ where}$$
$$f(x) = x^2 + 3x - 15$$

Functions can be defined as a **rule**.
Be sure to explicitly state what the
domain and codomain are!

$f : \mathbb{Z} \rightarrow \mathbb{N}$, where

$$f(n) = \begin{cases} n & \text{if } n \geq 0 \\ -n & \text{if } n \leq 0 \end{cases}$$

Some rules are given *piecewise*. We select which rule to apply based on the conditions on the right. (Just make sure at least one condition applies and that all applicable conditions give the same result!)

Some Nuances

Defining Functions

- Which of the following are functions?
 - $f : \mathbb{R} \rightarrow \mathbb{R}$ defined as : $f(x) = \frac{x+2}{x+1}$
 - $f : \mathbb{N} \rightarrow \mathbb{R}$ defined as : $f(x) = \frac{x+2}{x+1}$

Rules for defining functions:

$$\forall a \in A. \exists b \in B. f(a) = b$$

$$\forall a_1 \in A. \forall a_2 \in A. (a_1 = a_2 \rightarrow f(a_1) = f(a_2))$$

Defining Functions

- Which of the following are functions?

- $f : \mathbb{R} \rightarrow \mathbb{R}$ defined as $f(x) = \frac{x+2}{x+1}$

- $f : \mathbb{N} \rightarrow \mathbb{R}$ defined as

This expression isn't defined when $x = -1$, so f isn't defined over its full domain. We therefore don't consider it to be a function.

Rules for defining functions:

$$\forall a \in A. \exists b \in B. f(a) = b$$

$$\forall a_1 \in A. \forall a_2 \in A. (a_1 = a_2 \rightarrow f(a_1) = f(a_2))$$

Defining Functions

- Which of the following are functions?

- $f: \mathbb{R} \rightarrow \mathbb{R}$ defined as $f(x) = \frac{x+2}{x+1}$

- ✓ • $f: \mathbb{N} \rightarrow \mathbb{R}$ defined as $f(x) = \frac{x+2}{x+1}$

This is a function! Every natural number maps to some real number.

Rules for defining functions:

$$\forall a \in A. \exists b \in B. f(a) = b$$

$$\forall a_1 \in A. \forall a_2 \in A. (a_1 = a_2 \rightarrow f(a_1) = f(a_2))$$

Defining Functions

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Time-Out for Announcements!

Problem Set One Graded

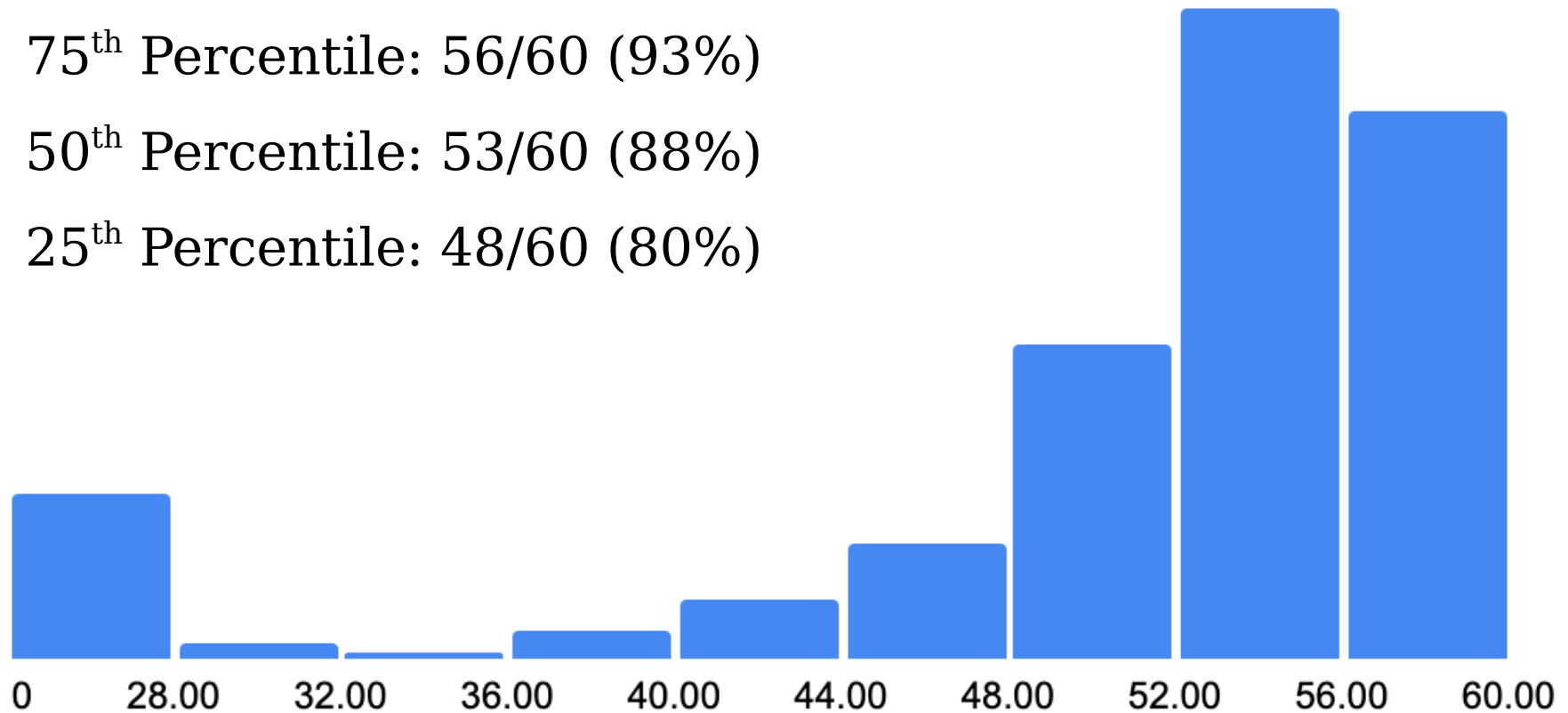
- Your wonderful TAs have finished grading Problem Set One.
- Grades and feedback are up on the Gradescope.
- Solutions are available online on the course website (visit the page for PS1 to get the link).

Problem Set 1 Graded

75th Percentile: 56/60 (93%)

50th Percentile: 53/60 (88%)

25th Percentile: 48/60 (80%)



Pro tips when reading a grading distribution:

1. Standard deviations are *malicious lies*. Ignore them.
2. The average score is a *malicious lie*. Ignore it.
3. Raw scores are *malicious lies*. Ignore them.

Regrade Requests

- We're human. We make mistakes. And we're happy to correct them!
- Regrades close one week after grades are released.
- Notes on regrades:
 - Please be civil. We make mistakes. We're happy to correct them.
 - We have to grade what you submitted; we can't take any clarifications into account during regrades.
 - Regrades are for where we made deductions we shouldn't have, rather than for the magnitude of deductions.

Back to CS103!

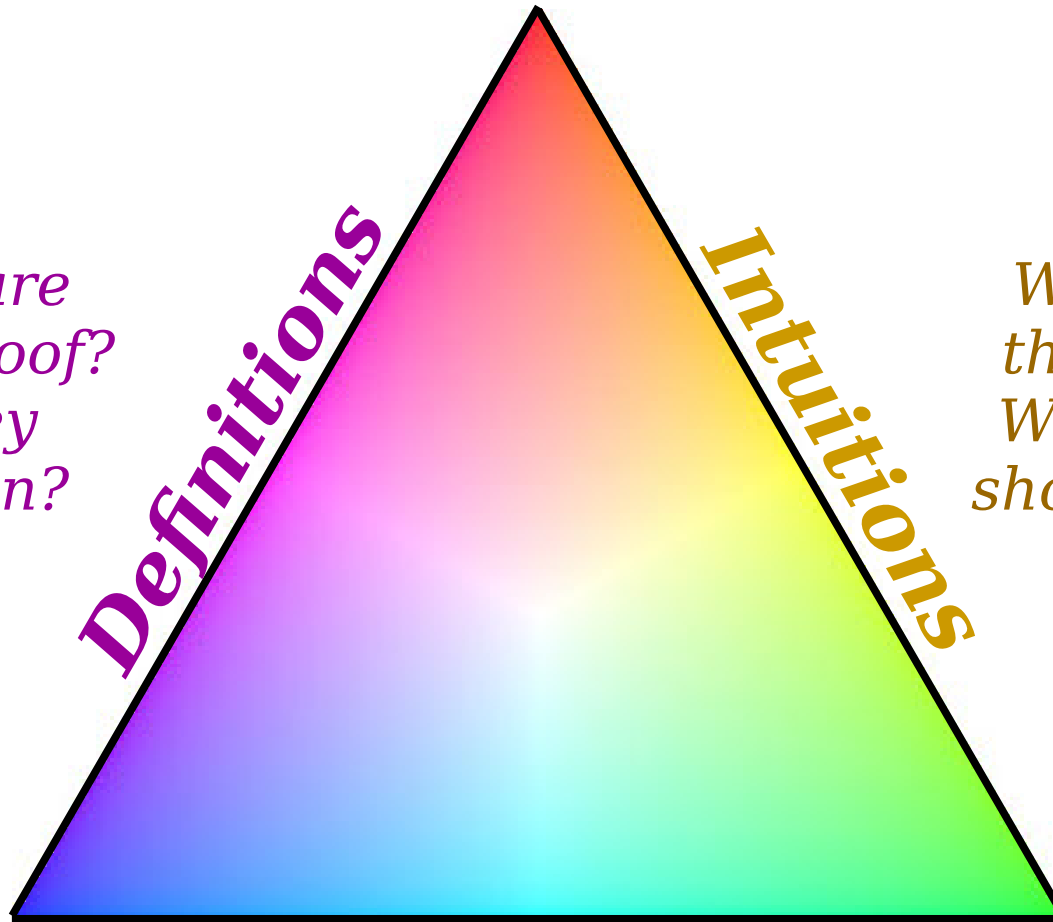
Special Types of Functions

*What terms are
used in this proof?
What do they
formally mean?*

Definitions

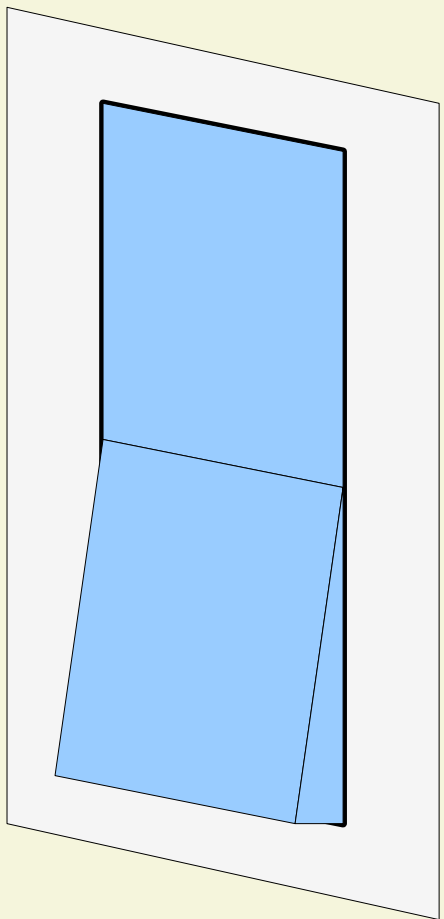
Intuitions

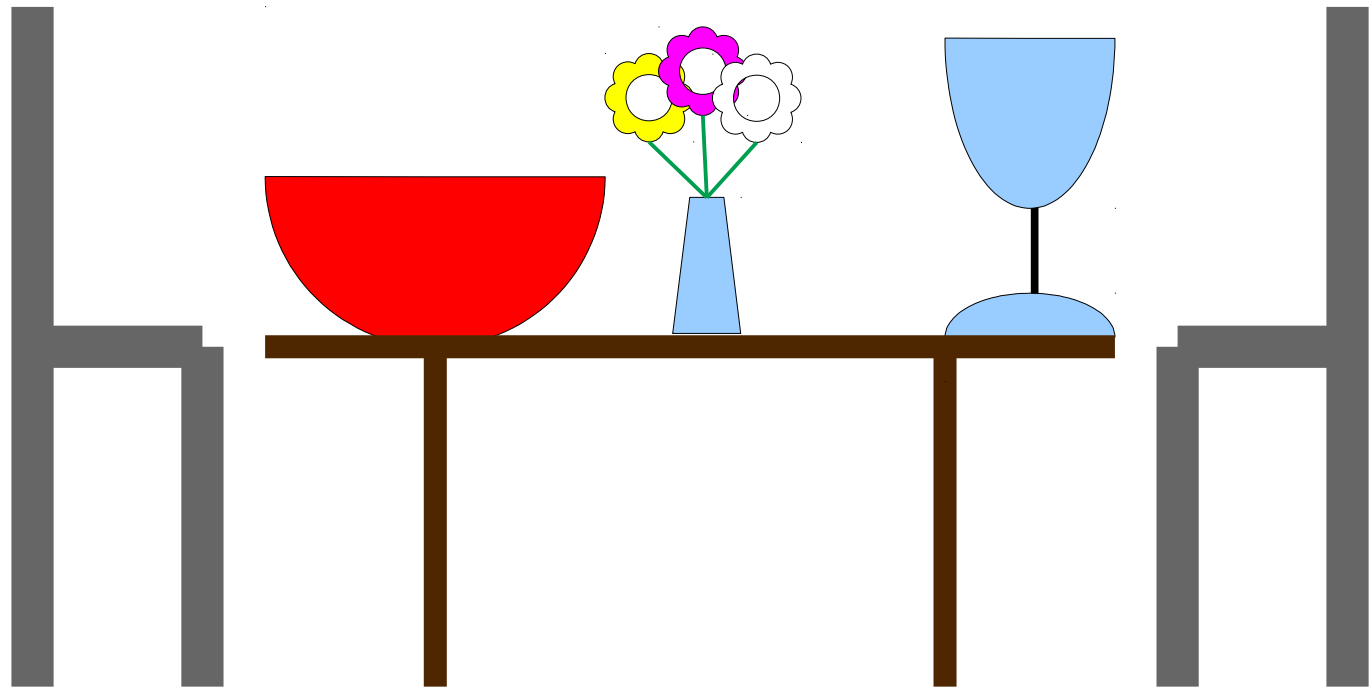
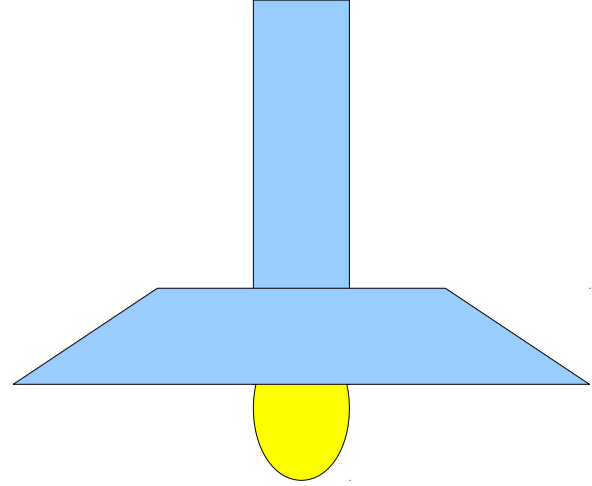
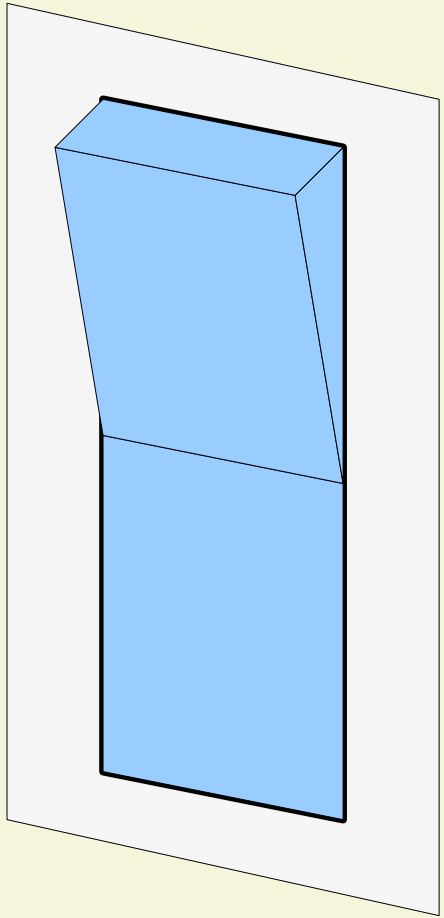
*What does this
theorem mean?
Why, intuitively,
should it be true?*



Conventions

*What is the standard
format for writing a proof?
What are the techniques
for doing so?*





Undoing by Doing Again

- Some operations invert themselves. For example:
 - Flipping a switch twice is the same as not flipping it at all.
 - In first-order logic, $\neg\neg A$ is equivalent to A .
 - In algebra, $-(-x) = x$.
 - In set theory, $(A \Delta B) \Delta B = A$. (*Yes, really!*)
- Operations with these properties are surprisingly useful in CS theory and come up in a bunch of contexts.
 - Storing compressed approximations of sets (XOR filters).
 - Building encryption systems (symmetric block ciphers).
 - Transmitting a large file to multiple receivers (fountain codes).

Involutions

- A function $f : A \rightarrow A$ from a set back to itself is called an ***involution*** if the following first-order logic statement is true about f :

$$\forall x \in A. f(f(x)) = x.$$

(“Applying f twice is equivalent to not applying f at all.”)

- Involutions have lots of interesting properties. Let’s explore them and see what we can find.

Involutions

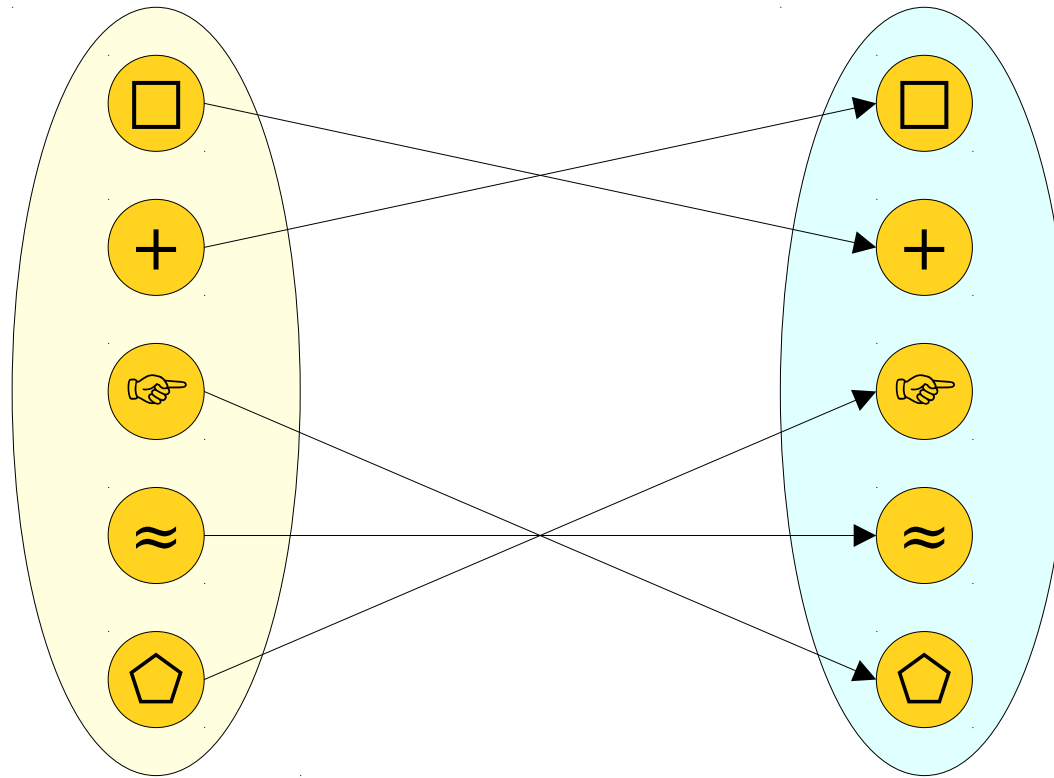
- Which of the following are involutions?
- ✓ • $f : \mathbb{Z} \rightarrow \mathbb{Z}$ defined as $f(x) = x$.
- ✓ • $g : \mathbb{Z} \rightarrow \mathbb{Z}$ defined as $g(x) = -x$.
- $h : \mathbb{R} \rightarrow \mathbb{R}$ defined as $h(x) = 1/x$. *Not a function!*
- ✓ • $p : \mathbb{N} \rightarrow \mathbb{N}$ defined as follows:

$$p(n) = \begin{cases} n+1 & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}$$

A function $f : A \rightarrow A$ is called an **involution** if the following first-order logic statement is true about f :

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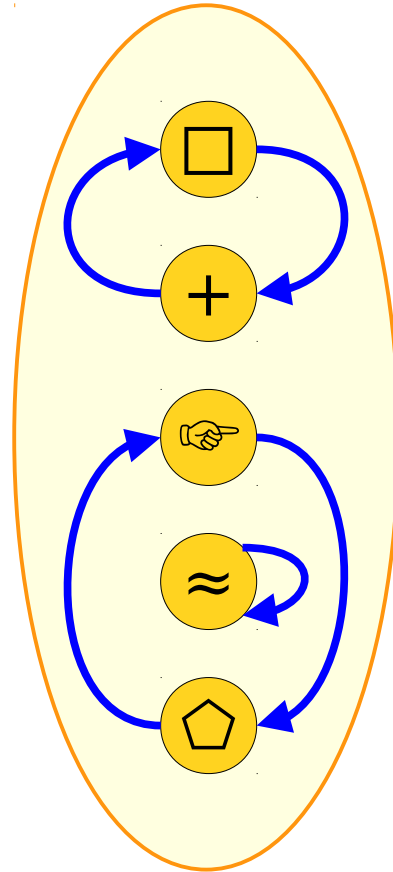
Involutions, Visually



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Involutions, Visually



A function $f : A \rightarrow A$ is called an ***involution*** if the following first-order logic statement is true about f :

$$\forall x \in A. f(f(x)) = x.$$

Proofs on Involutions

Theorem: The function $f : \mathbb{Z} \rightarrow \mathbb{Z}$ defined as

$$f(n) = \begin{cases} n+1 & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}$$

is an involution.

Proof:

What does it mean for f to be an involution?

$$\forall n \in \mathbb{Z}. f(f(n)) = n.$$

Therefore, we'll have the reader pick some $n \in \mathbb{Z}$, then argue that $f(f(n)) = n$.

Theorem: The function $f : \mathbb{Z} \rightarrow \mathbb{Z}$ defined as

$$f(n) = \begin{cases} n+1 & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}$$

is an involution.

Proof: Pick some $n \in \mathbb{Z}$. We need to show that $f(f(n)) = n$. To do so, we consider two cases.

Case 1: n is even. Then $f(n) = n+1$, which is odd. This means that $f(f(n)) = f(n+1) = (n+1) - 1 = n$.

Case 2: n is odd. Then $f(n) = n - 1$, which is even. Then we see that $f(f(n)) = f(n - 1) = (n - 1) + 1 = n$.

In either case, we see that $f(f(n)) = n$, which is what we need to show. ■

This proof contains no first-order logic syntax (quantifiers, connectives, etc.). It's written in plain English, just as usual.

Theorem: The function $f : \mathbb{N} \rightarrow \mathbb{N}$ defined as $f(n) = n^2$ is not an involution.

What does it mean for f to be an involution?

$$\forall n \in \mathbb{N}. f(f(n)) = n.$$

What is the negation of this statement?

$$\begin{aligned} &\neg \forall n \in \mathbb{N}. f(f(n)) = n \\ &\exists n \in \mathbb{N}. \neg (f(f(n)) = n) \\ &\exists n \in \mathbb{N}. f(f(n)) \neq n \end{aligned}$$

Therefore, we need to pick some concrete choice of n such that $f(f(n)) \neq n$.

Theorem: The function $f : \mathbb{N} \rightarrow \mathbb{N}$ defined as $f(n) = n^2$ is not an involution.

Proof: We need to show that there is some $n \in \mathbb{N}$ where $f(f(n)) \neq n$.

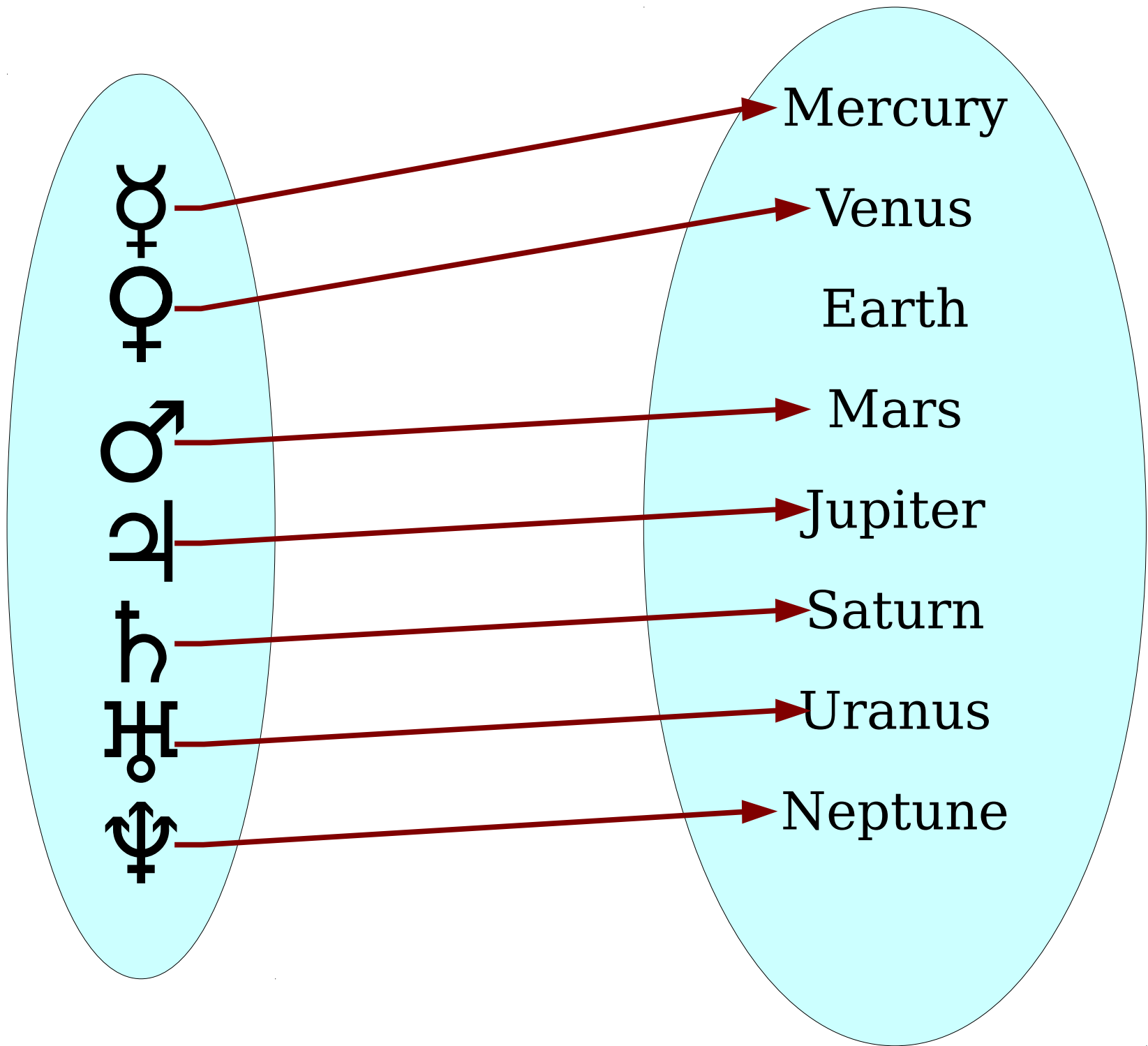
Pick $n = 2$. Then

$$\begin{aligned} f(f(n)) &= f(f(2)) \\ &= f(4) \\ &= 16, \end{aligned}$$

which means that $f(f(n)) \neq 2$, as required. ■

This proof contains no first-order logic syntax (quantifiers, connectives, etc.). It's written in plain English, just as usual.

Another Class of Functions



Injective Functions

- A function $f : A \rightarrow B$ is called **injective** (or **one-to-one**) if the following statement is true about f :

$$\forall a_1 \in A. \forall a_2 \in A. (a_1 \neq a_2 \rightarrow f(a_1) \neq f(a_2))$$

(“If the inputs are different, the outputs are different.”)

- The following first-order definition is equivalent (*why?*) and is often useful in proofs.

$$\forall a_1 \in A. \forall a_2 \in A. (f(a_1) = f(a_2) \rightarrow a_1 = a_2)$$

(“If the outputs are the same, the inputs are the same.”)

- A function with this property is called an **injection**.
- How does this compare to our second rule for functions?

Proofs on Injections

Injective Functions

Theorem: Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be defined as $f(n) = 2n + 7$.
Then f is injective.

Proof:

What does it mean for the function f to be injective?

$$\forall n_1 \in \mathbb{N}. \forall n_2 \in \mathbb{N}. (f(n_1) = f(n_2) \rightarrow n_1 = n_2)$$

$$\forall n_1 \in \mathbb{N}. \forall n_2 \in \mathbb{N}. (n_1 \neq n_2 \rightarrow f(n_1) \neq f(n_2))$$

Injective Functions

Theorem: Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be defined as $f(n) = 2n + 7$.
Then f is injective.

Proof: Consider any $n_1, n_2 \in \mathbb{N}$ where $f(n_1) = f(n_2)$. We will prove that $n_1 = n_2$.

Since $f(n_1) = f(n_2)$, we see that

$$2n_1 + 7 = 2n_2 + 7.$$

This in turn means that

$$2n_1 = 2n_2,$$

so $n_1 = n_2$, as required. ■

Good exercise: Repeat this proof using the other definition of injectivity!

Injective Functions

Theorem: Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be defined as $f(n) = 2n + 7$.
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$$2n_1 + 7 = 2n_2 + 7.$$

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$$2n_1 = 2n_2,$$

so $n_1 = n_2$, as required. ■

This proof contains no first-order logic syntax (quantifiers, connectives, etc.). It's written in plain English, just as usual.

Another Class of Functions

Lassen Peak

Mt. Shasta

Crater Lake

Mt. McLoughlin

Mt. Hood

Mt. St. Helens

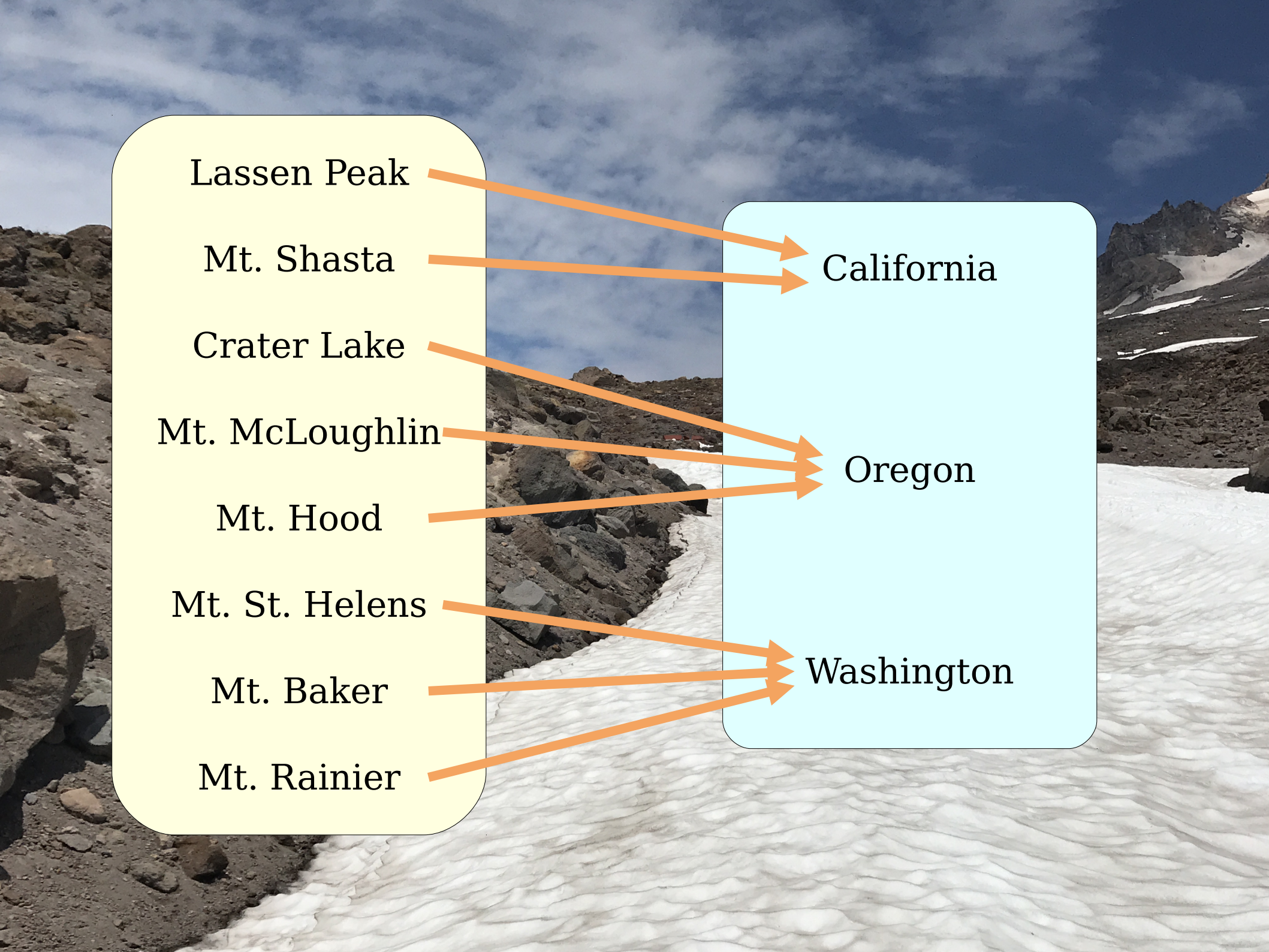
Mt. Baker

Mt. Rainier

California

Oregon

Washington



Surjective Functions

- A function $f : A \rightarrow B$ is called **surjective** (or **onto**) if this first-order logic statement is true about f :

$$\forall b \in B. \exists a \in A. f(a) = b$$

(“For every output, there's an input that produces it.”)

- A function with this property is called a **surjection**.
- How does this compare to our first rule of functions?

Next Time

- ***First-Order Assumptions***
 - The difference between assuming something is true and proving something is true.
- ***Connecting Function Types***
 - Involutions, injections, and surjections are related to one another. How?
- ***Function Composition***
 - Sequencing functions together.

Appendix: More Proofs on Functions

Injective Functions

Theorem: Let $f : \mathbb{Z} \rightarrow \mathbb{N}$ be defined as $f(x) = x^4$. Then f is not injective.

Proof: We will prove that there exist integers x_1 and x_2 such that $x_1 \neq x_2$, but $f(x_1) = f(x_2)$.

Let $x_1 = -1$ and $x_2 = +1$. Notice that

$$f(x_1) = f(-1) = (-1)^4 = 1$$

and

$$f(x_2) = f(1) = 1^4 = 1,$$

so $f(x_1) = f(x_2)$ even though $x_1 \neq x_2$, as required. ■

	To <i>prove</i> that this is true...	
$\forall x. A$	Have the reader pick an arbitrary x . We then prove A is true for that choice of x .	
$\exists x. A$	Find an x where A is true. Then prove that A is true for that specific choice of x .	
$A \rightarrow B$	Assume A is true, then prove B is true.	
$A \wedge B$	Prove A . Then prove B .	
$A \vee B$	Either prove $\neg A \rightarrow B$ or prove $\neg B \rightarrow A$. <i>(Why does this work?)</i>	
$A \leftrightarrow B$	Prove $A \rightarrow B$ and $B \rightarrow A$.	
$\neg A$	Simplify the negation, then consult this table on the result.	

Surjective Functions

Theorem: Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined as $f(x) = 2x$. Then $f(x)$ is surjective.

Proof: Consider any $y \in \mathbb{R}$. We will prove that there is a choice of $x \in \mathbb{R}$ such that $f(x) = y$.

Let $x = y / 2$. Then we see that

$$f(x) = f(y / 2) = 2y / 2 = y.$$

So we see that $f(x) = y$, as required. ■

Surjective Functions

Theorem: Let $g : \mathbb{N} \rightarrow \mathbb{N}$ be defined as $g(n) = 2n$. Then $g(x)$ is not surjective.

Proof: Let $n = 137$. Now, pick an arbitrary $m \in \mathbb{N}$. We need to show that $g(m) \neq n$.

Notice that $g(m) = 2m$ is even, while 137 is odd. Therefore, we have $g(m) \neq 137$, as required. ■